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ON A THEOREM OF F. SCHUR

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Let (M, g) be a C^4 Riemann manifold, $G_2(M)$ the Grassmann bundle of 2-planes on M, and $K: G_2(M) \to R$ the sectional curvature function. Let $\pi: G_2(M) \to M$ denote the canonical projection. Recall the theorem of F. Schur: if dimension $M \geq 3$, and $K|_{\pi^{-1}(p)} = \phi(p)$ for some $\phi: M \to R$, then (M, g) is of constant curvature. We shall view this theorem in the following setting:

Definition 1. Two Riemann manifolds (M, g), $(\bar{M}, \bar{g})$ are called *homocurved* if there exist a $1-1$ onto diffeomorphism $F: M \to \bar{M}$ and a function $\phi: M \to R$ such that for every $p \in M$ and $\sigma \in \pi^{-1}(p)$ we have

$$K(\sigma) = \phi(p)\bar{K}(F_*\sigma) ,$$

where $\bar{K}$ denotes the sectional curvature function of $(\bar{M}, \bar{g})$.

Definition 2. Homocurved manifolds are called homothetic (resp. *strongly homothetic*) if the corresponding $\phi \equiv$ constant (resp. F is a homothety).

'Strongly homothetic' clearly implies 'homothetic'. Converse is not true in general, e.g., consider the nonhomothetic conformal maps of constant curvature spaces. Schur's theorem says that a Riemann manifold of dimension ≥ 3 which is homocurved to a manifold of constant curvature is homothetic to it. A well-known fact about Einstein spaces is that a manifold homocurved to an Einstein manifold is homothetic to it.

Now we ask: *does "homocurved" imply "homothetic" in general*? We shall show that *generically* the answer to this question is yes.

Henceforth our standard situation will be the one described in Definition 1. *Throughout we shall use the notation and conventions of* [2].

Proposition 1. *Suppose that (M, g) is of dimension ≥ 3 and nowhere of constant curvature, i.e., on no nonempty open subset of M, $K \equiv$ constant. Then (M, g), $(\bar{M}, \bar{g})$ are conformal.*

Proof. This follows immediately from the general theorem of [2, § 2].

Proposition 2. *Suppose that (M, g) is of dimension ≥ 4 and nowhere conformally flat* (*cf.* [2, § 3]). *Then $\bar{R} = F_* R$, where $\bar{R}$ denotes the curvature tensor of $(\bar{M}, \bar{g})$.*

Proof. Identify M with $\bar{M}$ via F and consider the corresponding conformal deformation of the metric: $g \to F^*\bar{g} =$ (which we again denote by) $\bar{g} = f \cdot g$ where $f: M \to R$ is a positive real-value function. "Homocurved" implies

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$$\langle R(X, Y)X, Y\rangle = \frac{\psi}{f}\langle \bar{R}(X, Y)X, Y\rangle$$

for all vector fields X, Y. It easily follows that

$$R = \frac{\psi}{f}\bar{R}$$

(cf. [1, Proposition 3.1]).

Considering the conformal curvature tensor C and noting that it is a conformal invariant, we see that

$$\bar{C} = C = \frac{\psi}{f}\bar{C} .$$

Since (M, g) (and hence $(\bar{M}, \bar{g})$) is nowhere conformally flat of dimension ≥ 4, it follows from the well known theorem of Weyl that $\bar{C} \neq 0$ on a dense subset of M. So $\psi \equiv f$, and hence $\bar{R} = R$.

Corollary 1. *Under the hypothesis of the proposition, ψ is necessarily positive real-valued.*

We set $\psi = f = e^{2\phi}$, and use the notation of [2, § 7]). In particular, $G =$ grad ϕ, and $Q(X, Y) = XY\phi - (\nabla_X Y)\phi - X\phi Y\phi$.

Corollary 2. *Under the hypothesis of the proposition, for any vector field X on M we have*

$$Q(X, X) + \frac{\|X\|^2}{2}\|G\|^2 = 0 . \tag{1}$$

Proof. Since $\bar{R} = R$, we have $\bar{R} - R = T = 0$ (cf. [2, § 7]). Let X, Y, Z be mutually orthogonal. Then

$$0 = T(X, Y)Z = Q(Y, Z)X - Q(X, Z)Y .$$

It follows that for any two orthogonal vector fields X, Y, $Q(X, Y) = 0$. Hence, if $\|X\| = \|Y\|$, then $Q(X, X) = Q(Y, Y)$.

Let X, Y be orthogonal, and suppose that $\|X\| = \|Y\|$. Then

$$0 = \langle T(X, Y)X, Y\rangle = -\{Q(X, X) + Q(Y, Y) + \|X\|^2\|G\|^2\} ,$$

and Corollary 2 is now clear.

Theorem 1. *Let (M, g), $(\bar{M}, \bar{g})$ be homocurved, and suppose that (M, g) is complete, nowhere conformally flat and of dimension ≥ 4. Then (M, g), $(\bar{M}, \bar{g})$ are strongly homothetic.*

Proof. Since ϕ satisfies (1), as in [2, Proposition 10.1] we see that the trajectories of G are (pointsetwise) geodesics. By applying the argument of case i) in [2, Proposition 10.4], we thus obtain that $G \equiv 0$.

Despite this global result, it is clear however that even *locally*, *at least generically* the theorem ought to hold, which we now proceed to show.

Proposition 3. *Under the hypothesis of Proposition* 2, *suppose* $G_p \neq 0$ *at* $p \in M$. *Then for every* 2-*plane* σ *at* p *containing* G_p *we have* $K(\sigma) = 0$.

Proof. Let $\sum_{\mathrm{cycl}}$ denote the cyclic sum over X, Y, Z. Since $T = 0$, Proposition 7.7 of [2] implies that

$$\sum_{\mathrm{cycl}} \{\langle R(Y, Z)W, G\rangle X + \langle X, W\rangle R(Y, Z)G\} = 0 \; . \tag{2}$$

The argument of [2, § 9, Propositions 3 and 4] applied to (2) shows that there exists a constant c such that for any 2-plane σ at p containing G_p we have $K(\sigma) = c$. Now in (2) set $Y_p = W_p$, $Z_p = G_p/\|G_p\|$ and X_p, Y_p, Z_p to be orthonormal, and take inner product with X_p. We get

$$\langle R(Y_p, G_p)Y_p, G_p\rangle + \langle R(G_p, X_p)G_p, X_p\rangle = 0 \; ,$$

from this it clearly follows that $c = 0$. q.e.d.

The following theorem is now obvious:

Theorem 2. *Let* (M, g), $(\bar{M}, \bar{g})$ *be homocurved. Suppose that the dimension of* M *is* $n \geq 4$, *and that* (M, g) *is nowhere conformally flat. Then* (M, g), $(\bar{M}, \bar{g})$ *are strongly homothetic if*

(A) *The set* $\{p \in M \mid K|_{\pi^{-1}(p)}$ *does not take the value* $0\}$ *is dense in* M.

Remark. The condition (A) may be replaced by

(A′) The set $\{p \in M \mid$ if σ is a 2-plane at p such that $K(\sigma) = 0$, then σ is not a critical point of $K|_{\pi^{-1}(p)}$ of nullity $\geq n - 2\}$

is dense in M. This is due to the observations in [2, Theorem 9.5].

Remark. Instead of (A) we may impose certain analytic conditions under which Theorem 2 is valid. For instance, Proposition 3 shows that $R(X, G) = 0$ for any vector field X on M. *So Theorem* 2 *holds if* (A) *is replaced by*

(B) *The set* $\{p \in M \mid$ *There do not exist linearly independent* $X_p, Y_p \in T_p(M)$ *such that* $R(X_p, Y_p) = 0\}$

is dense in M.

Finally, we may impose some conditions on the diffeomorphism F. We have already seen $F_* R = \bar{R}$. In the spirit of Nomizu and Yano's formulation of the equivalence problem (cf. [3]) we contend: *Theorem* 2 *is valid if* (A) *is replaced by*

(C) $F_*(\nabla R) = \bar{\nabla}\bar{R}$, where ∇, $\bar{\nabla}$ denote the corresponding covariant derivations.

This condition is fulfilled, e.g., when M and $\bar{M}$ are symmetric spaces. Indeed, using Proposition 3 and [2], Proposition 7.6, we see that

$$0 = (\nabla_X R)(Y, Z)G - (\bar{\nabla}_X \bar{R})(Y, Z)G = \|G\|^2 R(Y, Z)X \; .$$

Hence, if $G \not\equiv 0$, then $R \equiv 0$ on a subset of M with nonempty interior; this contradicts the hypothesis that (M, g) is nowhere conformally flat.

References

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